Explicit Arithmetic for Modular Curves

Exercises II

(A) Let
$$E : Y^2 = X^3 + 2.$$
Let $P = (0, \sqrt{2}) \in E[3]$. Show that $[(E, P)] \in Y_1(3)(\mathbb{Q})$.

(B) Let
$$E : Y^2 = X^3 + 1.$$
Let $P = (\sqrt[3]{-4}, \sqrt{-3}) \in E[3]$. Show that $[(E, P)] \in Y_1(3)(\mathbb{Q})$.

(C) Let H be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ such that $-I \notin H$. Let E/K be an elliptic curve. Suppose, for all $\sigma \in G_K$ there is $h_\sigma \in H$ and $\phi_\sigma \in \{\pm I\}$ such that
$$\overline{\rho}_{E,N}(\sigma) = \phi_\sigma \cdot h_\sigma.$$
Show that $\sigma \mapsto \phi_\sigma$ is a character (i.e. a homomorphism).

Deduce that there is some quadratic twist E'/K such that $\overline{\rho}_{E',N}(G_K) \subset H$.